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QED_{1+1} by Dirac Quantization

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Abstract

Dirac quantization of electrodynamics on a two-dimensional cylindrical space-time, is considered in an explicit loop and functional representation. Expressions for a sufficient number of physical states are found, so that an inner product can be derived on the space of physical states. The relevant states are found to be a superposition of two disjoint sectors. The ground state is investigated and the investigations are met with partial success.

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1 Introduction and motivation

One of the main approaches towards a consistent quantization of gravity is Dirac quantization of general relativity formulated in Ashtekar's variables [1]. However, it appears to us that Dirac quantization of genuine, interacting field theories is a poorly investigated subject. This paper tries to fill that gap by considering a non-trivial model, electrodynamics on a $1+1$ -dimensional space-time, which can be thought of as a model for gravity coupled to fermions at the kinematical level. It can of course also be thought of as a model for electrodynamics or Yang-Mills coupled to fermions in $3+1$ -dimensions. The reason for choosing this particular model is that in the case of massless fermions, it was shown by Schwinger to be exactly solvable and hence the massless case is generally known as the Schwinger model. This model has been investigated in numerous papers over the years. Schwinger, however, considered space to be a real line. The Schwinger model on a circle has been considered in [3] and [4]. One of the crucial ingredients in the quantization of general relativity is the imposition of the so called "reality conditions", which ensures that the classical limit of the quantum theory has the correct reality properties. In the Ashtekar quantization program, the reality conditions are hoped to pick out the correct inner product. We find in this paper, that for the model investigated, the reality conditions suffice to pick out a "unique" inner product on the space of physical states. Another point of this paper is to avoid the use of any technique specific to two dimensions, like that of bosonization.

2 Hamiltonian formulation

Our starting point is the pseudo-classical Lagrangian density²

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \hbar\bar{\psi}\gamma^\mu(i\partial_\mu - eA_\mu)\psi - m\bar{\psi}\psi \quad (1)$$

(throughout we set $c = 1$ but keep \hbar for the sake of amusement), where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2)$$

$$\bar{\psi} = \psi^*\gamma^0 \quad (3)$$

and e and m are the coupling and mass constants respectively while $*$ denotes complex conjugation. A_μ is the EM-field and ψ is the (Grassmann odd) fermion field which both are taken to be $2\pi r$ -periodic. Furthermore we define $\gamma = \gamma^0\gamma^1$. An explicit representation of the gamma matrices is given by $\gamma^0 = \sigma_1$, $\gamma^1 = -i\sigma_2$ and $\gamma = \sigma_3$. The canonical procedure applied to the Lagrangian defined in (1) is described in detail in [5]. We will perform it here since it is of interest to us. In what follows, α, β, \dots , are taken as indices

²We use signature $(+, -)$ i.e. $\eta_{00} = -\eta_{11} = 1$, $[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}$, with $x^0 \equiv t$ (time) and $x^1 \equiv x \in [-\pi r, \pi r[$ (spatial coordinate on a circle with radius r).

labeling the different components of the spinors and the γ -matrices. Introducing the momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -F^{0\mu} \quad (4)$$

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_\alpha)} = -i\hbar\psi_\alpha^* \quad (5)$$

$$\pi_\alpha^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_\alpha^*)} = 0 \quad (6)$$

The non-vanishing Bose-Fermi brackets are,

$$\{A_\mu(x), \pi^\nu(y)\} = \delta_\mu^\nu \delta(x-y) \quad (7)$$

$$\{\psi_\alpha(x), \pi_\beta(y)\} = -\delta_{\alpha\beta} \delta(x-y) \quad (8)$$

$$\{\psi_\alpha^*(x), \pi_\beta^*(y)\} = -\delta_{\alpha\beta} \delta(x-y) \quad (9)$$

The Bose-Fermi brackets have the following properties:

$$\{A, B\} = -(-1)^{n_A n_B} \{B, A\} \quad (10)$$

$$\{A, B+C\} = \{A, B\} + \{A, C\} \quad (11)$$

$$\{A, BC\} = (-1)^{n_A n_B} B\{A, C\} + \{A, B\}C \quad (12)$$

$$\{AB, C\} = (-1)^{n_B n_C} \{A, C\}B + A\{B, C\} \quad (13)$$

where n is even or odd depending upon whether the function in question is Grassmann even or odd respectively. We get the primary constraints:

$$\phi = \pi^0 \approx 0 \quad (14)$$

$$\phi_\alpha = \pi_\alpha + i\hbar\psi_\alpha^* \approx 0 \quad (15)$$

$$\phi_\alpha^* = \pi_\alpha^* \approx 0 \quad (16)$$

The Hamiltonian density becomes:

$$\begin{aligned} \mathcal{H}_c &= \partial_0 A_\mu \pi^\mu + \partial_0 \psi_\alpha \pi_\alpha + \partial_0 \psi_\alpha^* \pi_\alpha^* - \mathcal{L} \\ &= \frac{1}{2}E^2 - \hbar\psi^* \gamma(i\partial_1 - eA)\psi + m\psi^* \gamma^0 \psi \\ &\quad - A_0 \chi + \partial_0 A_0 \phi + \partial_0 \psi_\alpha \phi_\alpha + \partial_0 \psi_\alpha^* \phi_\alpha^* \end{aligned} \quad (17)$$

where we've introduced $E = \pi^1$, $A = A_1$ and $\chi = \partial_1 E - e\hbar\psi_\alpha^* \psi_\alpha$. Hence we get the total Hamiltonian H_t :

$$H_t = \int dx \mathcal{H} + \int dx (u\phi + v_\alpha \phi_\alpha + w_\alpha \phi_\alpha^*) \quad (18)$$

where

$$\mathcal{H} = \frac{1}{2}E^2 - \hbar\psi^* \gamma(i\partial_1 - eA)\psi + m\psi^* \gamma^0 \psi - A_0 \chi, \quad (19)$$

and we've introduced Lagrange multipliers u, v_α and w_α , the latter two being Grassmann odd. Moving on to the constraint analysis we have:

$$\dot{\phi} = \{\phi, H_t\}|_{\phi=\phi_\alpha=\phi_\alpha^*=0} = \chi \approx 0 \quad (20)$$

$$\dot{\phi}_\alpha = \hbar(i\partial_1 + eA)(\psi^*\gamma)_\alpha + m(\psi^*\gamma^0)_\alpha + e\hbar A_0\psi_\alpha^* + i\hbar w_\alpha = 0 \quad (21)$$

$$\dot{\phi}_\alpha^* = \hbar(\gamma(i\partial_1 - eA)\psi)_\alpha - m(\gamma^0\psi)_\alpha - e\hbar A_0\psi_\alpha + i\hbar v_\alpha = 0 \quad (22)$$

which determines v_α and w_α . Hence we have single secondary constraint $\chi \approx 0$. Using (21) and (22) one finds $\dot{\chi} = 0$. Now let us consider the constraint algebra. We obviously have $0 = \{\phi, \phi_\alpha\} = \{\phi, \phi_\alpha^*\} = \{\phi, \chi\}$, and

$$\{\phi_\alpha(x), \phi_\beta^*(y)\} = -i\hbar\delta_{\alpha\beta}\delta(x-y) \quad (23)$$

$$\{\phi_\alpha(x), \chi(y)\} = -e\hbar\psi_\alpha^*(x)\delta(x-y) \quad (24)$$

$$\{\phi_\alpha^*(x), \chi(y)\} = e\hbar\psi_\alpha(x)\delta(x-y) \quad (25)$$

Letting $\varphi = \chi - ie(\phi_\alpha\psi_\alpha + \psi_\alpha^*\phi_\alpha^*)$ we find that its bracket with all the other constraints is weakly zero. Hence ϕ and φ are first class, while ϕ_α and ϕ_α^* are second class constraints. Thus

$$\begin{aligned} \Delta_{\alpha\beta}(x, y) &= \begin{bmatrix} \{\phi_\alpha(x), \phi_\beta(y)\} & \{\phi_\alpha(x), \phi_\beta^*(y)\} \\ \{\phi_\alpha^*(x), \phi_\beta(y)\} & \{\phi_\alpha^*(x), \phi_\beta^*(y)\} \end{bmatrix} \\ &= -i\hbar\delta(x-y) \begin{bmatrix} 0 & \delta_{\alpha\beta} \\ \delta_{\alpha\beta} & 0 \end{bmatrix} \end{aligned} \quad (26)$$

Then

$$\Delta_{\alpha\beta}^{-1} = \frac{i}{\hbar}\delta(x-y) \begin{bmatrix} 0 & \delta_{\alpha\beta} \\ \delta_{\alpha\beta} & 0 \end{bmatrix} \quad (27)$$

since

$$\int dz \Delta_{\alpha\gamma}(x, z) \Delta_{\gamma\beta}^{-1}(z, y) = \delta(x-y) \begin{bmatrix} \delta_{\alpha\beta} & 0 \\ 0 & \delta_{\alpha\beta} \end{bmatrix} \quad (28)$$

Therefore the Dirac bracket between two arbitrary phase space functions A and B becomes

$$\begin{aligned} \{A(x), B(y)\}_D &= \{A(x), B(y)\} \\ &- \int dz dw \{A(x), \begin{bmatrix} \phi_\alpha(z) \\ \phi_\alpha^*(z) \end{bmatrix}^T\} \Delta_{\alpha\beta}^{-1}(z, w) \{ \begin{bmatrix} \phi_\beta(w) \\ \phi_\beta^*(w) \end{bmatrix}, B(y) \} \\ &= \{A(x), B(y)\} \\ &- \int dz dw \{A(x), \phi_\alpha(z)\} \frac{i}{\hbar} \delta(w-z) \delta_{\alpha\beta} \{\phi_\beta^*(w), B(y)\} \\ &- \int dz dw \{A(x), \phi_\alpha^*(z)\} \frac{i}{\hbar} \delta(w-z) \delta_{\alpha\beta} \{\phi_\beta(w), B(y)\} \\ &= \{A(x), B(y)\} - \frac{i}{\hbar} \int dz \{A(x), \phi_\alpha(z)\} \{\phi_\alpha^*(z), B(y)\} \\ &- \frac{i}{\hbar} \int dz \{A(x), \phi_\alpha^*(z)\} \{\phi_\alpha(z), B(y)\} \end{aligned} \quad (29)$$

i.e.

$$\{\psi_\alpha(x), \psi_\beta^*(y)\}_D = \frac{1}{i\hbar} \delta_{\alpha\beta} \delta(x-y) \quad (30)$$

The bracket in (30) is the only fermionic bracket we'll need after having imposed the constraints $\phi_\alpha = \phi_\alpha^* = 0$. Imposing these two constraints we get

$$\pi_\alpha = -i\hbar\psi_\alpha^* \quad (31)$$

$$\pi_\alpha^* = 0 \quad (32)$$

Our first class constraint $\varphi \approx 0$ becomes

$$\varphi = \partial_1 E - e\hbar\psi_\alpha^*\psi_\alpha = \chi \approx 0 \quad (33)$$

which is just Gauss' law. Dropping the constraint $\phi \approx 0$ and instead considering $\lambda = A_0$ as a Lagrange multiplier leads to the reduced Hamiltonian $H = \int dx \mathcal{H}_{ph} - H_g$ where

$$\mathcal{H}_{ph} = \frac{1}{2}E^2 - \hbar\psi^*\gamma(i\partial_1 - eA)\psi + m\psi^*\gamma^0\psi \quad (34)$$

$$H_g = \int dx \lambda(x)\chi(x) \quad (35)$$

The only non-vanishing brackets are (dropping the subscript D):

$$\{A(x), E(y)\} = \delta(x-y) \quad (36)$$

$$\{\psi_\alpha(x), \psi_\beta^*(y)\} = \frac{1}{i\hbar} \delta_{\alpha\beta} \delta(x-y) \quad (37)$$

Let us finally consider infinitesimal gauge transformations, which are generated by the first class constraint H_g .

$$\{A(x), H_g\} = -\partial_x \lambda(x) \quad (38)$$

$$\{\psi_\alpha(x), H_g\} = ie\lambda(x)\psi_\alpha(x) \quad (39)$$

$$\{\psi_\alpha^*(x), H_g\} = -ie\lambda(x)\psi_\alpha^*(x) \quad (40)$$

Hence finite gauge transformations are of the form:

$$\begin{cases} A'(x) &= A(x) - \partial_x \lambda(x) \\ \psi'(x) &= e^{ie\lambda(x)}\psi(x) \\ \psi^{*'}(x) &= e^{-ie\lambda(x)}\psi^*(x) \end{cases} \quad (41)$$

2.1 Loop and line observables

Of particular interest when doing Dirac quantization are the physical, i.e. the gauge-invariant, functions on phase space called observables. Let

$$T(\gamma) = e^{ie \oint_\gamma dx A(x)} \quad (42)$$

where γ is a closed loop. The only loops on a circle are the ones that are labeled by the winding number n , i.e. they wind n times around the circle. Hence write

$$T(n) = e^{ie \oint_n dx A(x)} \quad (43)$$

Furthermore let

$$L_{\alpha\beta}(x, y) = \psi_\alpha^*(x) e^{ie \int_x^y dx' A(x')} \psi_\beta(y) \quad (44)$$

Using (41) it is easily shown that both T and $L_{\alpha\beta}$ are gauge-invariant³. The bracket algebra between observables takes the form

$$\{T(n), E(x)\} = ie \oint_n dx' \delta(x - x') T(n) = ien T(n) \quad (45)$$

$$\begin{aligned} \{L_{\alpha\beta}(x, y), E(z)\} &= \psi_\alpha^*(x) \psi_\beta(y) ie \int_x^y dx' \delta(x' - z) e^{ie \int_x^{x'} dx'' A(x'')} \\ &= ie \theta(x, y, z) L_{\alpha\beta}(x, y) \end{aligned} \quad (46)$$

where

$$\begin{aligned} \theta(x, y, z) &= \int_x^y dx' \delta(x' - z) = \frac{1}{2\pi r} \int_x^y dx' \sum_k e^{ik(x' - z)} \\ &= \frac{y - x}{2\pi r} + \frac{1}{2\pi r} \sum_{k \neq 0} \frac{1}{ik} (e^{ik(y - z)} - e^{ik(x - z)}) \end{aligned} \quad (47)$$

and k can take the values $k = 0, \pm \frac{1}{r}, \pm \frac{2}{r}, \dots$. Also,

$$\{T(n), L_{\alpha\beta}(x, y)\} = 0 \quad (48)$$

$$\begin{aligned} \{L_{\alpha\beta}(x, y), L_{\alpha'\beta'}(x', y')\} &= \frac{1}{i\hbar} \delta_{\beta\alpha'} \delta(y - x') L_{\alpha\beta'}(x, y') \\ &\quad - \frac{1}{i\hbar} \delta_{\alpha\beta'} \delta(x - y') L_{\alpha'\beta}(x', y) \end{aligned} \quad (49)$$

It turns out that it is useful to define $U(x, y) = e^{ie \int_x^y dx' A(x')}$ and hence

$$\begin{aligned} L_{\alpha\beta}(x, y) &= U(x, y) \psi_\alpha^*(x) \psi_\beta(y) \\ &= U(x, y) \frac{1}{2} (\psi_\alpha^*(x) \psi_\beta(y) - \psi_\beta(y) \psi_\alpha^*(x)) \end{aligned} \quad (50)$$

$$\{U(x, y), E(z)\} = ie \theta(x, y, z) U(x, y) \quad (51)$$

$$\{U(x, y), U(x', y')\} = 0 \quad (52)$$

We see that we can express the physical part of the Hamiltonian density, \mathcal{H}_{ph} , in terms of observables:

$$\mathcal{H}_{ph}(x) = \frac{1}{2} E^2(x) - i\hbar \lim_{y \rightarrow x} \gamma_{\alpha\beta} \partial_y L_{\alpha\beta}(x, y) + m \gamma_{\alpha\beta}^0 L_{\alpha\beta}(x, x) \quad (53)$$

³Here we allow both x and y to take any value on the real line.

and the first class constraint (33) as

$$\chi(x) = \partial_x E(x) - \rho(x) \approx 0 \quad (54)$$

where ρ is the charge density $\rho(x) = e\hbar L_{\alpha\alpha}(x, x)$. Also of interest is the momentum density, $\mathcal{P}(x)$,

$$\mathcal{P}(x) = -i\psi_\alpha^*(x)(i\partial_x - eA(x))\psi_\alpha(x)$$

i.e.

$$\mathcal{P}(x) = -i\hbar \lim_{y \rightarrow x} \partial_y L_{\alpha\alpha}(x, y). \quad (55)$$

Note that the electromagnetic field carries no momentum of its own. This is a consequence of the fact that pure electromagnetism in 1+1-dimensions has no local degrees of freedom.

3 Quantization

The general quantization program is described in [10] and [1]. We quantize the bracket algebra found in the previous section according to the rule

$$[\hat{A}, \hat{B}] = i\hbar \{\widehat{A, B}\} \quad (56)$$

for bosonic functions and

$$[\hat{A}, \hat{B}]_+ = i\hbar \{\widehat{A, B}\} \quad (57)$$

for fermionic. Hence quantization consists in finding a representation of the following commutator algebra:⁴

$$[\hat{T}(n), \hat{E}(x)] = -e\hbar n \hat{T}(n) \quad (58)$$

$$[\hat{L}_{\alpha\beta}(x, y), \hat{E}(z)] = -e\hbar \theta(x, y, z) \hat{L}_{\alpha\beta}(x, y) \quad (59)$$

$$\begin{aligned} [\hat{L}_{\alpha\beta}(x, y), \hat{L}_{\alpha'\beta'}(x', y')] &= \delta_{\beta\alpha'} \delta(y - x') \hat{L}_{\alpha\beta'}(x, y') \\ &- \delta_{\alpha\beta'} \delta(x - y') \hat{L}_{\alpha'\beta}(x', y) \end{aligned} \quad (60)$$

It turns out however, that it is difficult to find a “good”⁵ representation of the above observable algebra, so it seems that we have to represent non gauge-invariant operators as well (see however the end of this section). Instead of representing only the gauge-invariant operator $\hat{L}_{\alpha\beta}$ we represent $\hat{\psi}, \hat{\psi}^*$ and \hat{U} also. (The representation, of a similar algebra, found in [6] doesn’t seem to be completely satisfactory). Hence

$$[\hat{\psi}_\alpha(x), \hat{\psi}_\beta^*(y)]_+ = \delta_{\alpha\beta} \delta(x - y) \quad (61)$$

$$[\hat{U}(x, y), \hat{E}(z)] = -e\hbar \theta(x, y, z) \hat{U}(x, y) \quad (62)$$

⁴We denote commutators by $[\ , \]$ and anticommutators by $[\ , \]_+$.

⁵In the sense of having a representation space consisting only of gauge-invariant states.

should also be satisfied by the representation⁶. Then we define

$$\hat{L}_{\alpha\beta}(x, y) = \frac{1}{2}[\hat{\psi}_\alpha^*(x), \hat{\psi}_\beta(y)]\hat{U}(x, y) \quad (63)$$

Let us start out by representing the fermionic operators in (61). We use the representation first used in [7]. Hence let these operators act on wave functionals Ψ of a complex Grassmann field $\eta(x)$ by:

$$(\hat{\psi}_\alpha(x)\Psi)(\eta, \eta^*) = \frac{1}{\sqrt{2}}(\eta_\alpha(x) + \frac{\delta}{\delta\eta_\alpha^*(x)})\Psi(\eta, \eta^*) \quad (64)$$

$$(\hat{\psi}_\alpha^*(x)\Psi)(\eta, \eta^*) = \frac{1}{\sqrt{2}}(\eta_\alpha^*(x) + \frac{\delta}{\delta\eta_\alpha(x)})\Psi(\eta, \eta^*) \quad (65)$$

Proceeding to (58) we use the loop representation, introduced for gravity in [8] and for the pure Maxwell field in [9]. We also have to consider (62) simultaneously. Let the operators \hat{T} , \hat{E} and \hat{U} act on wave functionals of loops (an integer n in this case), and any number of coordinate pairs (x, y) representing the start and endpoints of a line. Then,

$$(\hat{T}(n)\Psi)(m, x_1, y_1, \dots, x_a, y_a) = \Psi(n + m, x_1, y_1, \dots, x_a, y_a) \quad (66)$$

$$(\hat{E}(x)\Psi)(m, x_1, y_1, \dots, x_a, y_a) = -e\hbar(n + \sum_{i=1}^a \theta(x_i, y_i, x)) \times \Psi(m, x_1, y_1, \dots, x_a, y_a) \quad (67)$$

$$(\hat{U}(x, y)\Psi)(m, x_1, y_1, \dots, x_a, y_a) = \Psi(m, x_1, y_1, \dots, x_a, y_a, x, y) \quad (68)$$

We also have

$$[\hat{U}(x, y), \hat{U}(x', y')] = 0 \quad (69)$$

and classically the U :s satisfy $U(x, z)U(z, y) = U(x, y)$, $U(x, x) = 1$ and

$$U(x, y + 2\pi r m) = U(x - 2\pi r m, y) = U(x, y)T(m).$$

We want these identities to be satisfied also quantum-mechanically. Hence by imposing the following restrictions on the wave functionals,

$$\begin{aligned} \Psi(n, x_1, y_1, \dots, x_i, y_i, \dots, x_j, y_j, \dots, x_a, y_a) &= \Psi(n, x_1, y_1, \dots, x_j, y_j, \dots, x_i, y_i, \dots, x_a, y_a) \\ \Psi(n, x_1, y_1, \dots, x_a, y_a, x, z, z, y) &= \Psi(n, x_1, y_1, \dots, x_a, y_a, x, y) \\ \Psi(n, x_1, y_1, \dots, x_a, y_a, x, x) &= \Psi(n, x_1, y_1, \dots, x_a, y_a) \\ \Psi(n, x_1, y_1, \dots, x_a, y_a + 2\pi r m) &= \Psi(n + m, x_1, y_1, \dots, x_a, y_a), \end{aligned} \quad (70)$$

(69) is satisfied and $\hat{U}(x, z)\hat{U}(z, y) = \hat{U}(x, y)$, $\hat{U}(x, x) = \hat{1}$,

$$\hat{U}(x, y + 2\pi r m) = \hat{U}(x - 2\pi r m, y) = \hat{U}(x, y)\hat{T}(m).$$

⁶We've only written the non-vanishing commutators and anticommutators, the vanishing ones should obviously also be satisfied.

Combining the two representations defined, i.e. writing $\Psi(n, x_1, y_1, \dots, x_a, y_a, \eta, \eta^*)$ and defining $\hat{L}_{\alpha\beta}$ by (63) one can easily check that (58)-(60) are satisfied, e.g. (58) (suppressing irrelevant coordinates)⁷:

$$\begin{aligned}
([\hat{T}(n), \hat{E}(x)]\Psi)(m) &= (\hat{T}(n)\hat{E}(x)\Psi)(m) - (\hat{E}(x)\hat{T}(n)\Psi)(m) \\
&= (\hat{T}(n)\Psi')(m) - (\hat{E}(x)\Psi'')(m) \\
&= \Psi'(m+n) - (-e\hbar m)\Psi''(m) \\
&= -e\hbar(m+n)\Psi(m+n) + e\hbar m\Psi(m+n) \\
&= -e\hbar n\Psi(m+n) = -e\hbar n(\hat{T}(n)\Psi)(m)
\end{aligned} \tag{71}$$

where $\Psi'(m) = -e\hbar m\Psi(m)$ and $\Psi''(m) = \Psi(m+n)$. The explicit action of $\hat{L}_{\alpha\beta}$ on a state is (by (63)):

$$\begin{aligned}
(\hat{L}_{\alpha\beta}(x, y)\Psi)(n, x_1, y_1, \dots, x_a, y_a, \eta, \eta^*) &= \\
\frac{1}{2}(\eta_\alpha^*(x)\eta_\beta(y) + \eta_\alpha^*(x)\frac{\delta}{\delta\eta_\beta^*(y)} - \eta_\beta(y)\frac{\delta}{\delta\eta_\alpha(x)} + \frac{\delta}{\delta\eta_\alpha(x)}\frac{\delta}{\delta\eta_\beta^*(y)}) \times \\
\Psi(n, x_1, y_1, \dots, x_a, y_a, x, y, \eta, \eta^*)
\end{aligned} \tag{72}$$

3.1 Pure Maxwell

An instructive example on the use of loop representations is the quantization of pure Maxwell i.e. we have the Hamiltonian operator

$$\hat{H} = \frac{1}{2} \int dx \hat{E}^2(x) \tag{73}$$

and wave functionals of loops $\Psi(n)$. We see that the Gauss' law constraint is trivially satisfied on loop states

$$(\partial_x \hat{E}(x)\Psi)(n) = \partial_x(-e\hbar n)\Psi(n) = 0 \tag{74}$$

Letting the Hamiltonian act one finds

$$(\hat{H}\Psi)(n) = \frac{1}{2} \int dx (-e\hbar n)^2 \Psi(n) = \pi r (e\hbar)^2 n^2 \Psi(n) \tag{75}$$

The eigenstates of \hat{H} are obviously the characteristic states $\Psi_m(n) = \delta_{m,n}$ with eigenvalues $E_m = \pi r (e\hbar m)^2$, $m = 0, \pm 1, \pm 2, \dots$. To find an inner product on this vector space we first consider the classical reality conditions:

$$E^*(x) = E(x) \tag{76}$$

$$T^*(n) = (e^{ie \oint_n dx A(x)})^* = T(-n) \tag{77}$$

⁷There are also terms $e\hbar \sum \theta(x_i, y_i, x)$ which trivially cancel, hence we don't bother to write them.

which, when quantized, turn into the hermitian adjoint relations:

$$\hat{E}^\dagger(x) = \hat{E}(x) \quad (78)$$

$$\hat{T}^\dagger(n) = \hat{T}(-n) \quad (79)$$

We use these relations, (78) and (79), to find an inner product. Try the following ansatz

$$\langle \Phi, \Psi \rangle = \sum_{n, n'=-\infty}^{\infty} \mu(n, n') \Phi^*(n) \Psi(n') \quad (80)$$

where the measure μ is positive definite and satisfies $\mu(n, n') = \mu(n', n)^*$. Hence μ should be chosen such that

$$\langle \Phi, \hat{E}(x) \Psi \rangle = \langle \hat{E}(x) \Phi, \Psi \rangle \quad (81)$$

$$\langle \Phi, \hat{T}(m) \Psi \rangle = \langle \hat{T}(-m) \Phi, \Psi \rangle \quad (82)$$

are satisfied. (81) leads to

$$\mu(n, n') = \mu(n) \delta_{n, n'} \quad (83)$$

and (82) implies

$$\mu(n, n' - m) = \mu(n + m, n') \quad (84)$$

Combining (83) and (84) we get

$$\mu(n) = \mu(n + m), \quad \forall m \quad (85)$$

and thus $\mu(n, n') = C \delta_{n, n'}$ for some real constant C , choose $C = 1$ for definiteness, i.e.

$$\langle \Phi, \Psi \rangle = \sum_{n=-\infty}^{\infty} \Phi^*(n) \Psi(n) \quad (86)$$

which also is positive definite. We see that the reality conditions suffice to pick out a “unique” inner product on the state space. Note further that the characteristic states Ψ_m form an ON-basis for the Hilbert space defined using the inner product (86).

3.2 Gauge-invariant states

Returning to the interacting theory we’ve come to the subject of gauge-invariant states. The gauge-invariant states are at the heart of Dirac quantization, all other states are considered irrelevant. The classical Gauss’ law constraint (54), when quantized should annihilate physical states i.e.

$$(\partial_x \hat{E}(x) - \hat{\rho}(x)) \Psi_{phys} = 0 \quad (87)$$

where $\hat{\rho}(x) = e\hbar \lim_{\epsilon \rightarrow 0} \hat{L}_{\alpha\alpha}(x, x+\epsilon)$. The idea to construct physical states is the following: first find one state, then by acting upon this state by the various physical operators we

can create new physical states. This first state should be as simple as possible. Call this state Ψ_{base} . Assume that it factorizes as:

$$\Psi_{base}(n, x_1, y_1, \dots, x_a, y_a, \eta, \eta^*) = \Psi_{base}(n, x_1, y_1, \dots, x_a, y_a) \Psi_{base}(\eta, \eta^*) \quad (88)$$

We solve (87) for Ψ_{base} by having it satisfy:

$$\partial_x \hat{E}(x) \Psi_{base} = 0 \quad (89)$$

$$\hat{\rho}(x) \Psi_{base} = 0 \quad (90)$$

(89) implies

$$\begin{aligned} & (\partial_x \hat{E}(x) \Psi_{base}) (n, x_1, y_1, \dots, x_a, y_a, \eta, \eta^*) = \\ & -e\hbar \partial_x (n + \sum_{i=1}^a \theta(x_i, y_i, x)) \Psi_{base}(n, x_1, y_1, \dots, x_a, y_a, \eta, \eta^*) = \\ & -e\hbar \sum_{i=1}^a (\delta(x - x_i) - \delta(x - y_i)) \Psi_{base}(n, x_1, y_1, \dots, x_a, y_a, \eta, \eta^*) = 0 \end{aligned} \quad (91)$$

i.e.

$$\sum_{i=1}^a (\delta(x - x_i) - \delta(x - y_i)) \Psi_{base}(n, x_1, y_1, \dots, x_a, y_a) = 0. \quad (92)$$

Hence $\Psi_{base}(n, x_1, y_1)$ can only have support on $x_1 + 2\pi r m = y_1$ for any integer m . Therefore let Ψ_{base} satisfy:

$$\Psi_{base}(n, x_1, y_1) = \begin{cases} \Psi_{base}(n + m) & \text{if } x_1 + 2\pi r m = y_1 \\ 0 & \text{otherwise} \end{cases} \quad (93)$$

and $\Psi_{base}(n, x_1, y_1, \dots, x_a, y_a)$ is non-zero only if it can be reduced to $\Psi_{base}(n + m, x_1, x_1)$ by using (70). Any such state can be written as a superposition of states $\Psi_{base}^{n_0}$ where $\Psi_{base}^{n_0}(n) = \delta_{n_0, n}$. Proceeding to (90) we make an ansatz for $\Psi_{base}(\eta, \eta^*)$:

$$\Psi_{base}(\eta, \eta^*) = \exp\left(\int dy \eta_\beta^*(y) M_{\beta\gamma} \eta_\gamma(y)\right) \quad (94)$$

where $M_{\beta\gamma}$ is a constant matrix. Hence we get (being very careful with the anticommutativity of η):

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \left(\hat{L}_{\alpha\alpha}(x, x + \epsilon) \Psi_{base} \right) (n, x_1, y_1, \dots, x_a, y_a, \eta, \eta^*) = \\ & \frac{1}{2} \lim_{\epsilon \rightarrow 0} (\eta_\alpha^*(x) \eta_\alpha(x + \epsilon) + M_{\alpha\alpha} \delta(\epsilon) - \eta_\alpha^*(x) M_{\alpha\beta} M_{\beta\gamma} \eta_\gamma(x + \epsilon)) \times \\ & \Psi_{base}(n, x_1, y_1, \dots, x_a, y_a, x, x + \epsilon, \eta, \eta^*) = 0. \end{aligned} \quad (95)$$

To have a well-defined limit we demand $M_{\alpha\alpha} = 0$ i.e. $\text{tr} M = 0$. Having done this we see that if $M^2 = 1$, then (95) is satisfied. Thus Ψ_{base} is gauge-invariant provided M is a

linear combination $a\gamma^0 + ib\gamma^1 + c\gamma$ such that $a^2 + b^2 + c^2 = 1$. Let us now consider some properties of $\Psi_{base}^{n_0}$. A calculation gives:

$$\hat{E}(z) \prod_{i=1}^a \hat{U}(x_i, y_i) \Psi_{base}^{n_0} = -e\hbar(n_0 + \sum_{i=1}^a \theta(y_i, x_i, z)) \prod_{j=1}^a \hat{U}(x_j, y_j) \Psi_{base}^{n_0}. \quad (96)$$

We will show this relation in the simplest case. First note (suppressing coordinates irrelevant for the calculation and for simplicity of notation restricting all coordinates to $[-\pi r, \pi r]$):

$$\begin{aligned} \left(\hat{U}(x_1, y_1) \Psi_{base}^{n_0} \right) (n, x, y) &= \Psi_{base}^{n_0}(n, x, y, x_1, y_1) = \\ &\begin{cases} \delta_{n_0, n} & \text{if } (x = y \text{ and } x_1 = y_1) \text{ or } (x = y_1 \text{ and } y = x_1) \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (97)$$

Hence

$$\begin{aligned} \left(\hat{E}(z) \hat{U}(x_1, y_1) \Psi_{base}^{n_0} \right) (n, x, y) &= -e\hbar(n + \theta(x, y, z)) \Psi_{base}^{n_0}(n, x, y, x_1, y_1) \\ &= -e\hbar(n_0 + \theta(y_1, x_1, z)) \Psi_{base}^{n_0}(n, x, y, x_1, y_1) \\ &= -e\hbar(n_0 + \theta(y_1, x_1, z)) \left(\hat{U}(x_1, y_1) \Psi_{base}^{n_0} \right) (n, x, y) \end{aligned} \quad (98)$$

since $\theta(x, x, z) = 0$. To completely specify the state $\Psi_{base}^{n_0}$ let us make the choices $M = \pm i\gamma^1$ (for reasons that will become apparent in a while). We can always, by suitable redefinitions of the gamma matrices, transform into those cases (we can obviously transform from $M = i\gamma^1$ to $M = -i\gamma^1$ as well but we choose not to). Now, acting with the physical operator $A_{\alpha\beta} \hat{L}_{\alpha\beta}$ on $\Psi_{base}^{n_0}$ we get a new gauge-invariant state (A is a matrix):

$$\begin{aligned} &\left(A_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) \Psi_{base}^{n_0} \right) (n, x_1, y_1, \dots, x_a, y_a, \eta, \eta^*) = \\ &(\eta_\alpha^*(x) \tilde{A}_{\alpha\beta} \eta_\beta(y) + \frac{1}{2} \text{tr}(MA) \delta(x - y)) \Psi_{base}^{n_0}(n, x_1, y_1, \dots, x_a, y_a, x, y, \eta, \eta^*) \end{aligned} \quad (99)$$

where $\tilde{A} = \frac{1}{2}(1 - M)A(1 + M)$. The tilde map has a number of important properties:

$$\begin{aligned} (1 - M)\tilde{A} &= 2\tilde{A} & \tilde{A}(1 + M) &= 2\tilde{A} \\ \tilde{A}(1 - M) &= 0 & (1 + M)\tilde{A} &= 0 \\ \tilde{A}_1 \tilde{A}_2 &= 0 & \text{tr} \tilde{A} &= 0 \\ \tilde{1} &= 0 & \tilde{M} &= 0, \end{aligned} \quad (100)$$

in general, and

$$\begin{aligned} \tilde{\gamma}^1 &= 0 & \tilde{\gamma}^0 &= \pm i\tilde{\gamma} \\ \frac{1}{2}(\tilde{\gamma}\gamma\tilde{\gamma}) &= \tilde{\gamma} & \frac{1}{2}\text{tr}(\gamma\tilde{\gamma}) &= 1 \\ \frac{1}{2}(\tilde{\gamma}\gamma^0\tilde{\gamma}) &= \mp i\tilde{\gamma} & \frac{1}{2}\text{tr}(\gamma^0\tilde{\gamma}) &= \mp i \end{aligned} \quad (101)$$

when $M = \pm i\gamma^1$. We see that the vector space of tilded matrices is one-dimensional and we choose $\tilde{\gamma}$ as a basis for these matrices. Now define:

$$\Psi_{(n_0)}^+ = \Psi_{base}^{n_0} \quad (102)$$

using $M = i\gamma^1$ i.e.

$$\begin{aligned} \Psi_{(n_0)}^+(n, x_1, y_1, \dots, x_a, y_a, \eta, \eta^*) = \\ \Psi_{base}^{n_0}(n, x_1, y_1, \dots, x_a, y_a) \exp\left(\int dx \eta_\alpha^*(x) i\gamma_{\alpha\beta}^1 \eta_\beta(x)\right) \end{aligned} \quad (103)$$

and similarly for $\Psi_{(n_0)}^-$ using $M = -i\gamma^1$. Inspired by the form of (99) we further define:

$$\begin{aligned} \Psi_{(n_0, x_1, y_1, \dots, x_a, y_a)}^+(n, x'_1, y'_1, \dots, x'_b, y'_b, \eta, \eta^*) = \\ \Psi_{base}^{n_0}(n, x_1, y_1, \dots, x_a, y_a, x'_1, y'_1, \dots, x'_b, y'_b) \times \\ \eta_{\alpha_1}^*(x_1) \tilde{\gamma}_{\alpha_1\beta_1} \eta_{\beta_1}(y_1) \cdots \eta_{\alpha_a}^*(x_a) \tilde{\gamma}_{\alpha_a\beta_a} \eta_{\beta_a}(y_a) \times \\ \exp\left(\int dx \eta_\alpha^*(x) i\gamma_{\alpha\beta}^1 \eta_\beta(x)\right) \end{aligned} \quad (104)$$

using $M = +i\gamma^1$ in the $\tilde{\gamma}$:s, and similarly define $\Psi_{(n_0, x_1, y_1, \dots, x_a, y_a)}^-$ using $M = -i\gamma^1$ everywhere. By (96) these states have the property:

$$\hat{E}(z) \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^+ = -e\hbar \left(n + \sum_{i=1}^a \theta(y_i, x_i, z)\right) \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^+ \quad (105)$$

$$\hat{E}(z) \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^- = -e\hbar \left(n + \sum_{i=1}^a \theta(y_i, x_i, z)\right) \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^-. \quad (106)$$

Checking gauge-invariance according to (87), they're indeed found to be gauge-invariant. Thus we've found a lot of physical states. Are they complete in the sense that they suffice to define an inner product on the space of physical states? To find an inner product we have to know the reality conditions of all operators acting on this space. The relevant operators are \hat{E} , $\hat{T}(m)$ and $\hat{L}_{\alpha\beta}$. As in the previous section we have $\hat{E}^\dagger(x) = \hat{E}(x)$ and $\hat{T}^\dagger(m) = \hat{T}(-m)$. Classically, $L_{\alpha\beta}$ satisfy:

$$L_{\alpha\beta}^*(x, y) = L_{\beta\alpha}(y, x). \quad (107)$$

Since a basis for all 2×2 matrices is given by $1, \gamma^0, \gamma^1$ and γ we get a basis for all $\hat{L}_{\alpha\beta}$:s by contracting with these matrices. Noting that $1, \gamma^0$ and γ are hermitian matrices while γ^1 is anti-hermitian we get the hermitian adjoint relations:

$$\begin{aligned} \hat{L}_{\alpha\alpha}^\dagger(x, y) &= \hat{L}_{\alpha\alpha}(y, x) & \gamma_{\alpha\beta}^0 \hat{L}_{\alpha\beta}^\dagger(x, y) &= \gamma_{\alpha\beta}^0 \hat{L}_{\alpha\beta}(y, x) \\ \gamma_{\alpha\beta}^1 \hat{L}_{\alpha\beta}^\dagger(x, y) &= -\gamma_{\alpha\beta}^1 \hat{L}_{\alpha\beta}(y, x) & \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}^\dagger(x, y) &= \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(y, x). \end{aligned} \quad (108)$$

It is not clear, at this stage, whether all these relations can be implemented simultaneously. That remains to be seen. Write $\mathcal{P} = \sum_M \oplus \mathcal{P}^M$, where \mathcal{P}^M denotes the space of states of the type $\Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^+$ having a fixed M . As a shorthand, write $\mathcal{P}^+ = \mathcal{P}^{+i\gamma^1}$ and $\mathcal{P}^- = \mathcal{P}^{-i\gamma^1}$. As we will show by construction, physical operators map \mathcal{P}^M into itself i.e. it forms an invariant subspace in the space of physical states \mathcal{P} . We will call the subspaces \mathcal{P}^M the different sectors of the theory. A general state in \mathcal{P}^+ can be written

$$\sum_{n=-\infty}^{\infty} \Psi_{f_n}^+,$$

where, (all integrals are taken from $-\pi r$ to πr),

$$\Psi_{f_n}^+ = f_n^{(0)} \Psi^+ + \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y f_n^{(a)}(x_1, y_1, \dots, x_a, y_a) \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^+, \quad (109)$$

and the functions $f_n^{(a)}$ are required to have the symmetry property:

$$f_n^{(a)}(x_1, y_1, \dots, x_i, y_i, \dots, x_j, y_j, \dots, x_a, y_a) = f_n^{(a)}(x_1, y_1, \dots, x_j, y_j, \dots, x_i, y_i, \dots, x_a, y_a). \quad (110)$$

Similarly define $\Psi_{f_n}^-$ in \mathcal{P}^- . By straightforward but tedious calculations, using (100) and (101), we've found the action of the various \hat{L} operators on these states. For \mathcal{P}^+ we have, (restricting x and y to $[-\pi r, \pi r]$),

$$\begin{aligned} \hat{L}_{\alpha\alpha}(x, y) \Psi_{f_n}^+ = & \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \sum_{b=1}^a \{ \delta(x - x_b) f_n^{(a)}(x_1, y_1, \dots, y, y_b, \dots, x_a, y_a) - \\ & \delta(y - y_b) f_n^{(a)}(x_1, y_1, \dots, x_b, x, \dots, x_a, y_a) \} \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^+ \end{aligned} \quad (111)$$

$$\begin{aligned} \gamma_{\alpha\beta}^0 \hat{L}_{\alpha\beta}(x, y) \Psi_{f_n}^+ = & -i f_n^{(1)}(y, x) \Psi_{(n)}^+ + \\ & i \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \{ \sum_{b=1}^a \{ f_n^{(a-1)}(x_1, y_1, \dots, \not{x}_b, \not{y}_b, \dots, x_a, y_a) \times \\ & \delta(x - x_b) \delta(y - y_b) \} - f_n^{(a+1)}(x_1, y_1, \dots, x_a, y_a, y, x) + \\ & \sum_{b=1}^a f_n^{(a+1)}(x_1, y_1, \dots, x_b, x, y, y_b, \dots, x_a, y_a) \} \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^+ \end{aligned} \quad (112)$$

$$\begin{aligned} \gamma_{\alpha\beta}^1 \hat{L}_{\alpha\beta}(x, y) \Psi_{f_n}^+ = & -i \delta(x - y) \Psi_{f_n}^+ + \\ & i \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \sum_{b=1}^a \{ \delta(x - x_b) f_n^{(a)}(x_1, y_1, \dots, y, y_b, \dots, x_a, y_a) + \\ & \delta(y - y_b) f_n^{(a)}(x_1, y_1, \dots, x_b, x, \dots, x_a, y_a) \} \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^+ \end{aligned} \quad (113)$$

$$\begin{aligned} \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) \Psi_{f_n}^+ = & f_n^{(1)}(y, x) \Psi_{(n)}^+ + \\ & \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \{ \sum_{b=1}^a \{ f_n^{(a-1)}(x_1, y_1, \dots, \not{x}_b, \not{y}_b, \dots, x_a, y_a) \times \\ & \delta(x - x_b) \delta(y - y_b) \} + f_n^{(a+1)}(x_1, y_1, \dots, x_a, y_a, y, x) - \\ & \sum_{b=1}^a f_n^{(a+1)}(x_1, y_1, \dots, x_b, x, y, y_b, \dots, x_a, y_a) \} \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^+ \end{aligned} \quad (114)$$

where $/$ denotes the absence of the indicated argument. Furthermore, by (105) we have

$$\begin{aligned} \hat{E}(x) \Psi_{f_n}^+ = & -e\hbar \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \sum_{b=1}^a \theta(y_b, x_b, x) f_n^{(a)}(x_1, y_1, \dots, x_a, y_a) \times \\ & \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^+. \end{aligned} \quad (115)$$

Also,

$$\hat{T}(m)\Psi_{f_n}^+ = f_n^{(0)}\Psi_{(n-m)}^+ + \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y f_n^{(a)}(x_1, y_1, \dots, x_a, y_a) \Psi_{(n-m, x_1, y_1, \dots, x_a, y_a)}^+. \quad (116)$$

The corresponding expressions for \mathcal{P}^- are found just by exchanging $i \leftrightarrow -i$ in the appropriate formula, hence:

$$\begin{aligned} \hat{L}_{\alpha\alpha}(x, y)\Psi_{f_n}^- = & \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \sum_{b=1}^a \{ \delta(x - x_b) f_n^{(a)}(x_1, y_1, \dots, y, y_b, \dots, x_a, y_a) - \\ & \delta(y - y_b) f_n^{(a)}(x_1, y_1, \dots, x_b, x, \dots, x_a, y_a) \} \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^- \end{aligned} \quad (117)$$

$$\begin{aligned} \gamma_{\alpha\beta}^0 \hat{L}_{\alpha\beta}(x, y)\Psi_{f_n}^- = & i f_n^{(1)}(y, x) \Psi_{(n)}^- - \\ & i \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \{ \sum_{b=1}^a \{ f_n^{(a-1)}(x_1, y_1, \dots, \cancel{x_b}, \cancel{y_b}, \dots, x_a, y_a) \times \\ & \delta(x - x_b) \delta(y - y_b) \} - f_n^{(a+1)}(x_1, y_1, \dots, x_a, y_a, y, x) + \\ & \sum_{b=1}^a f_n^{(a+1)}(x_1, y_1, \dots, x_b, x, y, y_b, \dots, x_a, y_a) \} \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^- \end{aligned} \quad (118)$$

$$\begin{aligned} \gamma_{\alpha\beta}^1 \hat{L}_{\alpha\beta}(x, y)\Psi_{f_n}^- = & i \delta(x - y) \Psi_{f_n}^- - \\ & i \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \sum_{b=1}^a \{ \delta(x - x_b) f_n^{(a)}(x_1, y_1, \dots, y, y_b, \dots, x_a, y_a) + \\ & \delta(y - y_b) f_n^{(a)}(x_1, y_1, \dots, x_b, x, \dots, x_a, y_a) \} \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^- \end{aligned} \quad (119)$$

$$\begin{aligned} \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y)\Psi_{f_n}^- = & f_n^{(1)}(y, x) \Psi_{(n)}^- + \\ & \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \{ \sum_{b=1}^a \{ f_n^{(a-1)}(x_1, y_1, \dots, \cancel{x_b}, \cancel{y_b}, \dots, x_a, y_a) \times \\ & \delta(x - x_b) \delta(y - y_b) \} + f_n^{(a+1)}(x_1, y_1, \dots, x_a, y_a, y, x) - \\ & \sum_{b=1}^a f_n^{(a+1)}(x_1, y_1, \dots, x_b, x, y, y_b, \dots, x_a, y_a) \} \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^- \end{aligned} \quad (120)$$

and

$$\begin{aligned} \hat{E}(x)\Psi_{f_n}^- = & -e\hbar \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \sum_{b=1}^a \theta(y_b, x_b, x) f_n^{(a)}(x_1, y_1, \dots, x_a, y_a) \times \\ & \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^-, \end{aligned} \quad (121)$$

$$\begin{aligned} \hat{T}(m)\Psi_{f_n}^- = & f_n^{(0)}\Psi_{(n-m)}^- + \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y f_n^{(a)}(x_1, y_1, \dots, x_a, y_a) \Psi_{(n-m, x_1, y_1, \dots, x_a, y_a)}^-. \end{aligned} \quad (122)$$

Thus we see explicitly that the various sectors $\mathcal{P}^+, \mathcal{P}^-, \dots$ are invariant under the action of the gauge-invariant operators. Now we can proceed to define an inner product. It is natural to require that the different sectors are orthogonal e.g.

$$< \Psi^+, \Psi^- > = 0 \quad (123)$$

Then for the $+$ sector we make the ansatz, using the experience from the last section, ($\Psi_{g_m}^+$ is defined similarly to $\Psi_{f_n}^+$):

$$\begin{aligned} < \Psi_{g_m}^+, \Psi_{f_n}^+ > = \\ & \delta_{n,m} \sum_{a,b} \int d^a x d^a y d^b x' d^b y' \mu(x_1, y_1, \dots, x_a, y_a, x'_1, y'_1, \dots, x'_b, y'_b) \times \\ & g_m^{(a)*}(x_1, y_1, \dots, x_a, y_a) f_n^{(b)}(x'_1, y'_1, \dots, x'_b, y'_b). \end{aligned} \quad (124)$$

Imposing $\hat{E}^\dagger(x) = \hat{E}(x)$ using (115) we get

$$\begin{aligned} \mu(x_1, y_1, \dots, x_a, y_a, x'_1, y'_1, \dots, x'_b, y'_b) = \\ \delta_{a,b} \mu(x_1, y_1, \dots, x_a, y_a, x'_1, y'_1, \dots, x'_a, y'_a). \end{aligned} \quad (125)$$

Imposing $\gamma_{\alpha\beta} \hat{L}_{\alpha\beta}^\dagger(x, y) = \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(y, x)$, i.e.

$$< \Psi_{g_m}^+, \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) \Psi_{f_n}^+ > = < \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(y, x) \Psi_{g_m}^+, \Psi_{f_n}^+ >,$$

we find by a tedious calculation using (114) that the measure μ is fully determined (apart from an arbitrary overall normalization factor), e.g.

$$\mu(x_1, y_1, x'_1, y'_1) = \delta(x_1 - x'_1) \delta(y_1 - y'_1)$$

and hence:

$$\begin{aligned} < \Psi_{g_m}^+, \Psi_{f_n}^+ > = \delta_{n,m} (g_n^{(0)*} f_n^{(0)} + \\ & \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \epsilon_{i_1 \dots i_a} g_n^{(a)*}(x_1, y_{i_1}, \dots, x_a, y_{i_a}) \times \\ & f_n^{(a)}(x_1, y_1, \dots, x_a, y_a)) \end{aligned} \quad (126)$$

where $\epsilon_{i_1 \dots i_a}$ is the totally antisymmetric symbol. Checking the remaining adjoint relations in (108) we find that they are all satisfied by the inner product (126). Similarly for the $-$ sector we get an analogous result:

$$\begin{aligned} < \Psi_{g_m}^-, \Psi_{f_n}^- > = \delta_{n,m} (g_n^{(0)*} f_n^{(0)} + \\ & \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \epsilon_{i_1 \dots i_a} g_n^{(a)*}(x_1, y_{i_1}, \dots, x_a, y_{i_a}) \times \\ & f_n^{(a)}(x_1, y_1, \dots, x_a, y_a)). \end{aligned} \quad (127)$$

3.3 Cyclic representation

Having, with much labour, calculated all gauge-invariant states in the functional representation it is easy to see that these states can be constructed in a much simpler, algebraic manner. Introduce the operators,

$$\begin{aligned}\hat{L}_+(x, y) &= \frac{1}{2}(\gamma - i\gamma^0)_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) & \hat{L}_-(x, y) &= \frac{1}{2}(\gamma + i\gamma^0)_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) \\ \hat{K}_+(x, y) &= \frac{1}{2}(1 - i\gamma^1)_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) & \hat{K}_-(x, y) &= \frac{1}{2}(1 + i\gamma^1)_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y).\end{aligned}$$

Using (60), we obtain,

$$\begin{aligned}[\hat{L}_+(x, y), \hat{L}_+(x', y')] &= [\hat{L}_-(x, y), \hat{L}_-(x', y')] = 0 \\ [\hat{L}_-(x, y), \hat{L}_+(x', y')] &= \delta(y - x') \hat{K}_-(x, y') - \delta(x - y') \hat{K}_+(x', y) \\ [\hat{K}_-(x, y), \hat{K}_-(x', y')] &= \delta(y - x') \hat{K}_-(x, y') - \delta(x - y') \hat{K}_-(x', y) \\ [\hat{K}_+(x, y), \hat{K}_+(x', y')] &= \delta(y - x') \hat{K}_+(x, y') - \delta(x - y') \hat{K}_+(x', y) \\ [\hat{K}_+(x, y), \hat{K}_-(x', y')] &= 0 \\ [\hat{K}_-(x, y), \hat{L}_-(x', y')] &= \delta(x' - y) \hat{L}_-(x, y') \\ [\hat{K}_-(x, y), \hat{L}_+(x', y')] &= -\delta(x - y') \hat{L}_+(x', y) \\ [\hat{K}_+(x, y), \hat{L}_+(x', y')] &= \delta(x' - y) \hat{L}_+(x, y') \\ [\hat{K}_+(x, y), \hat{L}_-(x', y')] &= -\delta(x - y') \hat{L}_-(x', y).\end{aligned}\tag{128}$$

Also, by (58) and (59),

$$\begin{aligned}[\hat{T}(n), \hat{E}(x)] &= -e\hbar n \hat{T}(n) \\ [\hat{L}(x, y), \hat{E}(z)] &= -e\hbar \theta(x, y, z) \hat{L}(x, y),\end{aligned}\tag{129}$$

where in the last equation, \hat{L} can stand for any one of the operators \hat{L}_+ , \hat{L}_- , \hat{K}_+ or \hat{K}_- . Furthermore, we have the relations,

$$\hat{T}(n)\hat{T}(m) = \hat{T}(n + m)$$

and

$$\hat{L}(x, y + 2\pi r m) = \hat{L}(x - 2\pi r m, y) = \hat{L}(x, y) \hat{T}(m).$$

Now introduce the cyclic states $\Psi_{(0)}^+$ and $\Psi_{(0)}^-$ with the properties

$$< \Psi_{(0)}^+, \Psi_{(0)}^+ > = < \Psi_{(0)}^-, \Psi_{(0)}^- > = 1, \quad < \Psi_{(0)}^+, \Psi_{(0)}^- > = 0.$$

In what follows, restrict x and y to the interval $[-\pi r, \pi r[$,

$$\begin{aligned}\hat{E}(x)\Psi_{(0)}^+ &= 0, \quad \hat{L}_-(x, y)\Psi_{(0)}^+ = 0, \quad \hat{T}(m)\Psi_{(0)}^+ = \Psi_{(-m)}^+, \quad \hat{L}_+(x, y)\Psi_{(0)}^+ = \Psi_{(0, x, y)}^+ \\ \hat{E}(x)\Psi_{(0)}^- &= 0, \quad \hat{L}_+(x, y)\Psi_{(0)}^- = 0, \quad \hat{T}(m)\Psi_{(0)}^- = \Psi_{(-m)}^-, \quad \hat{L}_-(x, y)\Psi_{(0)}^- = \Psi_{(0, x, y)}^- \\ \hat{K}_+(x, y)\Psi_{(0)}^+ &= -\frac{1}{2}\delta(x - y)\Psi_{(0)}^+, \quad \hat{K}_-(x, y)\Psi_{(0)}^+ = \frac{1}{2}\delta(x - y)\Psi_{(0)}^+ \\ \hat{K}_-(x, y)\Psi_{(0)}^- &= -\frac{1}{2}\delta(x - y)\Psi_{(0)}^-, \quad \hat{K}_+(x, y)\Psi_{(0)}^- = \frac{1}{2}\delta(x - y)\Psi_{(0)}^-.\end{aligned}$$

Furthermore,

$$\begin{aligned}\hat{T}(-n) \prod_{j=1}^a \hat{L}_+(x_j, y_j) \Psi_{(0)}^+ &= \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^+, \\ \hat{T}(-n) \prod_{j=1}^a \hat{L}_-(x_j, y_j) \Psi_{(0)}^- &= \Psi_{(n, x_1, y_1, \dots, x_a, y_a)}^-, \end{aligned}$$

and

$$\begin{aligned}\hat{L}_+^\dagger(x, y) &= \hat{L}_-(y, x), & \hat{K}_+^\dagger(x, y) &= \hat{K}_+(y, x), & \hat{K}_-^\dagger(x, y) &= \hat{K}_-(y, x), \\ \hat{T}^\dagger(n) &= \hat{T}(-n), & \hat{E}^\dagger(x) &= \hat{E}(x).\end{aligned}$$

Using the algebra, defined in (128) and (129), we can easily reproduce all results found in the previous subsection, including the inner product.

4 Dynamics

Until now, it hasn't really been important what the Hamiltonian looks like. The discussion in the previous section about physical states applies to any theory with a Hamiltonian having the same symmetries (and the same field content), i.e. being constrained by Gauss' law (87). But we're interested in the specific Hamiltonian which is given classically by (53) and hence we write

$$\hat{H} = \int dx \left(\frac{1}{2} \hat{E}^2(x) - i\hbar \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial(x + \epsilon)} \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, x + \epsilon) + m \lim_{\epsilon \rightarrow 0} \gamma_{\alpha\beta}^0 \hat{L}_{\alpha\beta}(x, x + \epsilon) \right). \quad (130)$$

\hat{H} contains the two operators $\gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, x + \epsilon)$ and $\gamma_{\alpha\beta}^0 \hat{L}_{\alpha\beta}(x, x + \epsilon)$. If we let these operators act on the state $\Psi_{base}^{n_0}$ defined in the previous section we see, by (99), that if we are to have a finite limit when ϵ tends to zero, we must demand $\text{tr}(M\gamma) = 0$ and $\text{tr}(M\gamma^0) = 0$. These two conditions, together with the previously found conditions $\text{tr}M = 0$ and $M^2 = 1$, implies that $M = \pm i\gamma^1$ (which is the reason why we've already studied these cases). Hence, of all the sectors of physical states available, we rule out all except for the $+$ and $-$ sector when considering the specific dynamics governed by the Hamiltonian (130), i.e. physical states are in general a superposition of states from the $+$ and $-$ sector. To solve for the dynamics, we essentially have to find a complete set of eigenstates to the Hamiltonian. The momentum operator, which is given classically by (55), is

$$\hat{P} = -i\hbar \int dx \lim_{\epsilon \rightarrow 0} \hat{L}_{\alpha\alpha}(x, x + \epsilon). \quad (131)$$

Since \hat{H} and \hat{P} commute, they can be simultaneously diagonalized. Thus we label states by their energy and momentum eigenvalues. Let us try to find the ground state. Denote it by Ω and write $\Omega = C_+ \Omega^+ + C_- \Omega^-$ where Ω^+ and Ω^- belongs to the $+$ and $-$ sector respectively (C_+ and C_- are constants). The ground state satisfies:

$$\hat{H}\Omega = E_{(0)}\Omega \quad (132)$$

$$\hat{P}\Omega = 0, \quad (133)$$

where $E_{(0)}$ denotes the vacuum energy. Note that there is no way we can “normal order” the Hamiltonian prior to solving for Ω to get rid of $E_{(0)}$. We simply have to live with the vacuum energy. After having solved for Ω we can effectively discard it. The appearance of $E_{(0)}$ in (132) implies that not only the ground state satisfies (132) and (133), but also a whole range of other states. We will see further on how to pick out the correct one(s). We make the ansatz

$$\Omega_{n_0}^+ = \Psi_{G_+}^+ \quad (134)$$

$$\Omega_{n_0}^- = \Psi_{G_-}^- \quad (135)$$

with support on $n = n_0$ and where $G_+^{(0)} = G_-^{(0)} = 1$, i.e. (by (109))

$$\Omega_{n_0}^+ = \Psi_{(n_0)}^+ + \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y G_+^{(a)}(x_1, y_1, \dots, x_a, y_a) \Psi_{(n_0, x_1, y_1, \dots, x_a, y_a)}^+ \quad (136)$$

$$\Omega_{n_0}^- = \Psi_{(n_0)}^- + \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y G_-^{(a)}(x_1, y_1, \dots, x_a, y_a) \Psi_{(n_0, x_1, y_1, \dots, x_a, y_a)}^-. \quad (137)$$

Now we're ready to consider (132) for $\Omega_{n_0}^+$ and $\Omega_{n_0}^-$. We will first do a simple case in detail.

$$\hat{E}(x) \Psi_{(n_0, x_1, y_1)}^+ = -e\hbar(n_0 + \theta(y_1, x_1, x)) \Psi_{(n_0, x_1, y_1)}^+$$

and hence,

$$\hat{E}(x)^2 \Psi_{(n_0, x_1, y_1)}^+ = (e\hbar)^2 (n_0^2 + 2n_0\theta(y_1, x_1, x) + \theta(y_1, x_1, x)^2) \Psi_{(n_0, x_1, y_1)}^+,$$

i.e.

$$\frac{1}{2} \int dx \hat{E}^2(x) \Psi_{(n_0, x_1, y_1)}^+ = \frac{(e\hbar)^2}{2} (2\pi r n_0^2 - 2n_0(y_1 - x_1) + V(x_1, y_1)) \Psi_{(n_0, x_1, y_1)}^+$$

where the potential V is given by (using (47))

$$V(x_1, y_1) = \int dx \theta(x_1, y_1, x)^2 = \frac{(y_1 - x_1)^2}{2\pi r} + p(x_1 - y_1) \quad (138)$$

$$p(x_1 - y_1) = \frac{1}{\pi r} \sum_{k \neq 0} \frac{1}{k^2} (1 - e^{ik(x_1 - y_1)}). \quad (139)$$

Furthermore define

$$W_{n_0}(x_1, y_1) = V(x_1, y_1) - 2n_0(y_1 - x_1). \quad (140)$$

Note that $V(x_1, y_1) = |x_1 - y_1|$ as long as $-\pi r < x_1, y_1 < \pi r$. In general we get

$$\begin{aligned} \frac{1}{2} \int dx \hat{E}^2(x) \Omega_{n_0}^+ &= \frac{(e\hbar)^2}{2} \{ 2\pi r n_0^2 \Psi_{(n_0)}^+ + \\ &\sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y (2\pi r n_0^2 + W_{n_0}(x_1, y_1, \dots, x_a, y_a)) \times \\ &G_+^{(a)}(x_1, y_1, \dots, x_a, y_a) \Psi_{(n_0, x_1, y_1, \dots, x_a, y_a)}^+ \} \end{aligned} \quad (141)$$

where

$$W_{n_0}(x_1, y_1, \dots, x_a, y_a) = V(x_1, y_1, \dots, x_a, y_a) - 2n_0(y_1 - x_1 + \dots + y_a - x_a) \quad (142)$$

and

$$\begin{aligned} V(x_1, y_1, \dots, x_a, y_a) &= \frac{1}{2\pi r}(y_1 - x_1 + \dots + y_a - x_a)^2 + \sum_{i,j=1}^a p(x_i - y_j) \\ &- \sum_{j>i=1}^a (p(x_i - x_j) + p(y_i - y_j)). \end{aligned} \quad (143)$$

Hence, using (112) and (114), we get

$$\begin{aligned} \hat{H}\Omega_{n_0}^+ &= ((e\hbar)^2\pi r n_0^2 - i \int dx \lim_{\epsilon \rightarrow 0} (\hbar \partial_{x+\epsilon} + m) G_+^{(1)}(x + \epsilon, x)) \Psi_{(n_0)}^+ + \\ &\sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \left\{ \frac{(e\hbar)^2}{2} (2\pi r n_0^2 + W_{n_0}(x_1, y_1, \dots, x_a, y_a)) \times \right. \\ &G_+^{(a)}(x_1, y_1, \dots, x_a, y_a) + \\ &i \sum_{b=1}^a G_+^{(a-1)}(x_1, y_1, \dots, \not{x}_b, \not{y}_b, \dots, x_a, y_a) \left(\hbar \frac{\partial}{\partial y_b} + m \right) \delta(y_b - x_b) - \\ &i \int dx \lim_{\epsilon \rightarrow 0} (\hbar \partial_{x+\epsilon} + m) G_+^{(a+1)}(x_1, y_1, \dots, x_a, y_a, x + \epsilon, x) + \\ &i \int dx \lim_{\epsilon \rightarrow 0} (\hbar \partial_{x+\epsilon} + m) \sum_{b=1}^a G_+^{(a+1)}(x_1, y_1, \dots, x_b, x, x + \epsilon, y_b, \dots, x_a, y_a) \left. \right\} \times \\ &\Psi_{(n_0, x_1, y_1, \dots, x_a, y_a)}^+ = E_{(0)} \Omega_{n_0}^+. \end{aligned} \quad (144)$$

Thus we can identify,

$$E_{(0)} = \pi r (e\hbar n_0)^2 - i \int dx \lim_{\epsilon \rightarrow 0} (\hbar \partial_{x+\epsilon} + m) G_+^{(1)}(x + \epsilon, x) \quad (145)$$

and

$$\begin{aligned} &-i \int dx \lim_{\epsilon \rightarrow 0} (\hbar \partial_{x+\epsilon} + m) \sum_{b=1}^a G_+^{(a+1)}(x_1, y_1, \dots, x_b, x, x + \epsilon, y_b, \dots, x_a, y_a) + \\ &i \int dx \lim_{\epsilon \rightarrow 0} (\hbar \partial_{x+\epsilon} + m) G_+^{(a+1)}(x_1, y_1, \dots, x_a, y_a, x + \epsilon, x) = \\ &i \int dx \lim_{\epsilon \rightarrow 0} (\hbar \partial_{x+\epsilon} + m) G_+^{(1)}(x + \epsilon, x) G_+^{(a)}(x_1, y_1, \dots, x_a, y_a) - \\ &i \sum_{b=1}^a G_+^{(a-1)}(x_1, y_1, \dots, \not{x}_b, \not{y}_b, \dots, x_a, y_a) (\hbar \partial_{x_b} - m) \delta(x_b - y_b) + \\ &\frac{(e\hbar)^2}{2} W_{n_0}(x_1, y_1, \dots, x_a, y_a) G_+^{(a)}(x_1, y_1, \dots, x_a, y_a). \end{aligned} \quad (146)$$

Continuing on to (133) and using (111) we find

$$\begin{aligned}\hat{P}\Omega_{n_0}^+ = & \\ & -i\hbar\Psi_{n_0}\sum_{a=1}^{\infty}\frac{1}{a!}\int d^axd^ay\sum_{b=1}^a(\partial_{x_b}+\partial_{y_b})G_+^{(a)}(x_1,y_1,\dots,x_a,y_a)\times \\ & \Psi_{(n_0,x_1,y_1,\dots,x_a,y_a)}^+ = 0\end{aligned}$$

i.e.

$$\sum_{b=1}^a(\partial_{x_b}+\partial_{y_b})G_+^{(a)}(x_1,y_1,\dots,x_a,y_a)=0$$

and hence $G_+^{(a)}$ is translationally invariant, i.e.

$$G_+^{(a)}(x_1+d,y_1+d,\dots,x_a+d,y_a+d)=G_+^{(a)}(x_1,y_1,\dots,x_a,y_a) \quad (147)$$

for any displacement d . All in all, (145)-(147) determine the $+$ sector component of the ground state. (145) and (146) are not really well defined as they stand, because they involve the evaluation of a distribution at zero. Hence, we take it as being understood, that an ultraviolet cutoff has been performed to regularize these equations. Similarly, the corresponding expressions for the $-$ sector are found by letting $m \leftrightarrow -m$. Thus,

$$E_{(0)} = \pi r(e\hbar n_0)^2 - i \int dx \lim_{\epsilon \rightarrow 0} (\hbar \partial_{x+\epsilon} - m) G_-^{(1)}(x+\epsilon, x), \quad (148)$$

$$\begin{aligned}& -i \int dx \lim_{\epsilon \rightarrow 0} (\hbar \partial_{x+\epsilon} - m) \sum_{b=1}^a G_-^{(a+1)}(x_1, y_1, \dots, x_b, x, x+\epsilon, y_b, \dots, x_a, y_a) + \\ & i \int dx \lim_{\epsilon \rightarrow 0} (\hbar \partial_{x+\epsilon} - m) G_-^{(a+1)}(x_1, y_1, \dots, x_a, y_a, x+\epsilon, x) = \\ & i \int dx \lim_{\epsilon \rightarrow 0} (\hbar \partial_{x+\epsilon} - m) G_-^{(1)}(x+\epsilon, x) G_-^{(a)}(x_1, y_1, \dots, x_a, y_a) - \\ & i \sum_{b=1}^a G_-^{(a-1)}(x_1, y_1, \dots, \cancel{x_b}, \cancel{y_b}, \dots, x_a, y_a) (\hbar \partial_{x_b} + m) \delta(x_b - y_b) + \\ & \frac{(e\hbar)^2}{2} W_{n_0}(x_1, y_1, \dots, x_a, y_a) G_-^{(a)}(x_1, y_1, \dots, x_a, y_a) \quad (149)\end{aligned}$$

and

$$G_-^{(a)}(x_1+d,y_1+d,\dots,x_a+d,y_a+d)=G_-^{(a)}(x_1,y_1,\dots,x_a,y_a). \quad (150)$$

Hence we see that $\Omega_{n_0}^+$ and $\Omega_{n_0}^-$ are coupled only through the vacuum energy $E_{(0)}$.

4.1 Vacuum and VEV's when $e = 0$

Before embarking on a solution attempt to the interacting vacuum equations (145)-(150), let us study the special case $e = 0$. Then the potential term drops out from (146) and

(149). Write $g_+(x_1, y_1) = G_+^{(1)}(x_1, y_1)$ and $g_-(x_1, y_1) = G_-^{(1)}(x_1, y_1)$. It is easily seen that a solution to all the vacuum equations is given as (independently of n_0),

$$g_+(x_1, y_1) = -\frac{1}{2\pi r} \sum_k e^{ik(x_1 - y_1)} \frac{(\hbar k + im)}{E_k} \quad (151)$$

$$g_-(x_1, y_1) = -\frac{1}{2\pi r} \sum_k e^{ik(x_1 - y_1)} \frac{(\hbar k - im)}{E_k} \quad (152)$$

where $E_k = \sqrt{(\hbar k)^2 + m^2}$ and

$$G_+^{(a)}(x_1, y_1, \dots, x_a, y_a) = g_+(x_1, y_1) g_+(x_2, y_2) \cdots g_+(x_a, y_a) \quad (153)$$

$$G_-^{(a)}(x_1, y_1, \dots, x_a, y_a) = g_-(x_1, y_1) g_-(x_2, y_2) \cdots g_-(x_a, y_a), \quad (154)$$

i.e.

$$\begin{aligned} \Omega_{n_0}^+ &= \exp \left(\int dx_1 dy_1 g_+(x_1, y_1) \hat{L}_+(x_1, y_1) \right) \Psi_{(n_0)}^+ \\ \Omega_{n_0}^- &= \exp \left(\int dx_1 dy_1 g_-(x_1, y_1) \hat{L}_-(x_1, y_1) \right) \Psi_{(n_0)}^- \end{aligned} \quad (155)$$

The true groundstate when $e = 0$ should be independent of the electromagnetic degrees of freedom, hence consider the states

$$\Omega^+ = \sum_{n_0=-\infty}^{\infty} \Omega_{n_0}^+,$$

and

$$\Omega^- = \sum_{n_0=-\infty}^{\infty} \Omega_{n_0}^-.$$

This solution really is the ground state in the case $e = 0$ even though we will postpone the proof of this until later. The expression for $E_{(0)}$ is, for both the $+$ and $-$ sector using (145) and (148),

$$E_{(0)} = - \sum_k E_k.$$

The cutoff is performed by allowing Fourier modes up to $|k| = \frac{N}{r}$ where N is some integer. Note the following identities

$$\int dx g_+^*(x, y) g_+(x, y') = \frac{1}{2\pi r} \sum_k e^{ik(y - y')} = \delta(y - y') \quad (156)$$

$$\int dx g_-^*(x, y) g_-(x, y') = \delta(y - y') \quad (157)$$

and

$$g_+^*(x_1, y_1) = g_-(y_1, x_1). \quad (158)$$

Also,

$$\int dx_1 dy_1 g_+^*(x_1, y_1) g_+(x_1, y_1) = 2\pi r \delta(0) = 2N + 1 = \lambda,$$

where we've introduced $\lambda = 2N + 1$. We further note that the cutoff delta functions work exactly like ordinary delta functions provided they are integrated against functions that are similarly cut off. With this knowledge, we proceed to calculate some vacuum expectation values. Hence using (126), (dropping the sum over n_0 which just factors out),

$$\begin{aligned} \langle \Omega^+, \Omega^+ \rangle &= 1 + \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \epsilon_{i_1 \dots i_a} g_+^*(x_1, y_{i_1}) \cdots g_+^*(x_a, y_{i_a}) \times \\ &\quad g_+(x_1, y_1) \cdots g_+(x_a, y_a) = \\ &= 1 + \sum_{a=1}^{\infty} \frac{1}{a!} \int d^a y \epsilon_{i_1 \dots i_a} \delta(y_{i_1} - y_1) \cdots \delta(y_{i_a} - y_a) = \\ &= 1 + \sum_{a=1}^{\infty} p_a(\lambda) = 2^\lambda \quad (\lambda \geq 0), \end{aligned} \tag{159}$$

where

$$p_a(\lambda) = \frac{1}{a!} \prod_{i=0}^{a-1} (\lambda - i). \tag{160}$$

In a similar manner

$$\langle \Omega^-, \Omega^- \rangle = 2^\lambda.$$

Hence, by (123),

$$\langle \Omega, \Omega \rangle = 2^\lambda (|C_+|^2 + |C_-|^2) \tag{161}$$

where $\Omega = C_+ \Omega^+ + C_- \Omega^-$. Next let us consider the expectation value of the operator $\gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y)$. Using (114) we get

$$\begin{aligned} \langle \Omega^+, \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) \Omega^+ \rangle &= g_+(y, x) 2^\lambda + \\ &\quad (g_-(y, x) - g_+(y, x)) \sum_{a=1}^{\infty} a \frac{p_a(\lambda)}{\lambda} = \\ &= g_+(y, x) 2^\lambda + (g_-(y, x) - g_+(y, x)) 2^{\lambda-1} \quad (\lambda \geq 1). \end{aligned} \tag{162}$$

Similarly,

$$\begin{aligned} \langle \Omega^-, \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) \Omega^- \rangle &= \\ &= g_-(y, x) 2^\lambda + (g_+(y, x) - g_-(y, x)) 2^{\lambda-1} \quad (\lambda \geq 1). \end{aligned} \tag{163}$$

Hence,

$$\begin{aligned} \langle \Omega, \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) \Omega \rangle &= |C_+|^2 (g_+(y, x) 2^\lambda + (g_-(y, x) - g_+(y, x)) 2^{\lambda-1}) + \\ &\quad |C_-|^2 (g_-(y, x) 2^\lambda + (g_+(y, x) - g_-(y, x)) 2^{\lambda-1}). \end{aligned} \tag{164}$$

Something very odd has happened. By (108) we should have

$$(\gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y))^\dagger = \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(y, x)$$

i.e.

$$< \Omega, \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) \Omega >^* = < \Omega, \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(y, x) \Omega >,$$

but

$$\begin{aligned} < \Omega, \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) \Omega >^* = |C_+|^2 (g_-(x, y) 2^\lambda + (g_+(x, y) - g_-(x, y)) 2^{\lambda-1}) + \\ & |C_-|^2 (g_+(x, y) 2^\lambda + (g_-(x, y) - g_+(x, y)) 2^{\lambda-1}), \end{aligned} \quad (165)$$

since $g_+^*(x, y) = g_-(y, x)$. Hence for consistency we must demand $|C_+| = |C_-|$ i.e. C_+ and C_- are equal up to an irrelevant phase. Thus, choosing

$$C_+ = C_- = 1$$

i.e. $\Omega = \Omega^+ + \Omega^-$, we find

$$< \Omega, \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) \Omega > = (g_+(y, x) + g_-(y, x)) 2^\lambda, \quad (166)$$

and hence, using (161),

$$\begin{aligned} \frac{< \Omega, \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, y) \Omega >}{< \Omega, \Omega >} &= \frac{1}{2} (g_+(y, x) + g_-(y, x)) \\ &= -\frac{1}{2\pi r} \sum_k e^{ik(y-x)} \frac{\hbar k}{E_k}. \end{aligned} \quad (167)$$

We note that the end result is independent of the cutoff N , as it should be. What has happened? In a previous subsection we constructed an inner product that was to have all the correct reality properties, yet when we use it here it doesn't have the right properties unless $|C_+| = |C_-|$. The only reasonable explanation of this mystery, is that somehow the distributional character of g_+ and g_- with its regularization, ruined the (formal) derivation. One can also argue that Ω^+ and Ω^- are improper states (not normalizable) and we cannot expect the adjoint relations to hold on arbitrary such states. Furthermore, we note that something had to go wrong, since otherwise we wouldn't have obtained a definite answer for the expectation value. At this stage we can only speculate about what happens when $e \neq 0$, but the apparent symmetry between the $+$ and $-$ sector seems to suggest that this is a general result, i.e. $\Omega_{n_0} = \Omega_{n_0}^+ + \Omega_{n_0}^-$ even when $e \neq 0$. Similarly, we find

$$\begin{aligned} \frac{< \Omega, \gamma_{\alpha\beta}^0 \hat{L}_{\alpha\beta}(x, y) \Omega >}{< \Omega, \Omega >} &= \frac{i}{2} (g_-(y, x) - g_+(y, x)) \\ &= -\frac{1}{2\pi r} \sum_k e^{ik(y-x)} \frac{m}{E_k} \end{aligned} \quad (168)$$

and

$$\frac{< \Omega, \gamma_{\alpha\beta}^1 \hat{L}_{\alpha\beta}(x, y) \Omega >}{< \Omega, \Omega >} = 0 \quad (169)$$

$$\frac{< \Omega, \hat{L}_{\alpha\alpha}(x, y) \Omega >}{< \Omega, \Omega >} = 0. \quad (170)$$

All these results agree with

$$\langle 0 | \frac{1}{2} [\hat{\psi}_\alpha^\dagger(x), \hat{\psi}_\beta(y)] | 0 \rangle = -\frac{1}{4\pi r} \sum_k e^{ik(y-x)} \frac{(\hbar k \gamma + m \gamma^0)_{\alpha\beta}}{E_k} \quad (171)$$

obtained using ordinary free quantization of fermions. This is encouraging. Even though we have required gauge invariance in all cases, i.e. even in the case $e = 0$, we get the same result as the one one gets having set $e = 0$ from the outset and then of course not worrying about gauge invariance. Note also that had we calculated the expectation value of these operators on the state Ω_{n_0} , we would have obtained the same results as above, as long as x and y were in the interval $[-\pi r, \pi r]$. For values of x and y outside this interval the expectation value is zero for all \hat{L} operators.

4.2 Solution attempt to the vacuum equations

Let us try to solve (145)-(150) in the general case. The solution should, in the limit $e \rightarrow 0$ approach the solution found in the previous section. This is sufficient for the solution to be the vacuum (or a vacuum in the case of vacuum degeneracy). By looking at the form of the potential W_{n_0} in (142) and ignoring the two non-periodic terms that depends on $y_1 - x_1 + \dots y_a - x_a$, we make the ansatz (being for the moment only concerned about the $+$ sector),

$$G_+^{(a)}(x_1, y_1, \dots, x_a, y_a) = g_+(x_1, y_1) \cdots g_+(x_a, y_a) \phi^{(a)}(x_1, y_1, \dots, x_a, y_a) \quad (172)$$

where

$$\phi^{(a)}(x_1, y_1, \dots, x_a, y_a) = \exp \left\{ \sum_{i,j=1}^a \varphi(x_i - y_j) - \sum_{j>i=1}^a (\varphi(x_i - x_j) + \varphi(y_i - y_j)) \right\}, \quad (173)$$

and we demand $\varphi(0) = 0$, $\varphi(x) = \varphi(-x)$ which implies $\varphi'(0) = 0$. Furthermore, we let φ be a $2\pi r$ -periodic function. We have not been able to generalize this ansatz (in any good way) to accommodate a functional dependence on $y_1 - x_1 + \dots$. We will anyway insert it into the vacuum equations, just dropping the non-periodic terms in W_{n_0} . This might be a valid short distance approximation. We note that the ansatz satisfies (147). Let us discuss some properties of the function $\phi^{(a)}$. We have

$$\phi^{(a+1)}(x_1, y_1, \dots, x_a, y_a, x, x) = \phi^{(a)}(x_1, y_1, \dots, x_a, y_a) \quad (174)$$

$$\phi^{(a+1)}(x_1, y_1, \dots, x_b, x, x, y_b, \dots, x_a, y_a) = \phi^{(a)}(x_1, y_1, \dots, x_a, y_a) \quad (175)$$

and

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \partial_{x+\epsilon} \phi^{(a+1)}(x_1, y_1, \dots, x_a, y_a, x + \epsilon, x) = \\ \phi^{(a)}(x_1, y_1, \dots, x_a, y_a) \sum_{i=1}^a (\varphi'(x - y_i) - \varphi'(x - x_i)) \end{aligned} \quad (176)$$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \partial_{x+\epsilon} \phi^{(a+1)}(x_1, y_1, \dots, x_b, x, x + \epsilon, y_b, \dots, x_a, y_a) = \\ \phi^{(a)}(x_1, y_1, \dots, x_a, y_a) \sum_{i=1}^a (\varphi'(x - y_i) - \varphi'(x - x_i)). \end{aligned} \quad (177)$$

Let us next consider (146). Using the properties of the $e = 0$ solution we find that (146) becomes

$$\begin{aligned}
& -i\hbar \sum_{b=1}^a \prod_{j \neq b} g_+(x_j, y_j) \{ \delta(y_b - x_b) \sum_{i=1}^a (\varphi'(y_i - y_b) - \varphi'(x_i - y_b)) + \\
& \int dx g_+(x_b, x) g_+(x, y_b) \sum_{i=1}^a (\varphi'(x - y_i) - \varphi'(x - x_i)) \} = \\
& \frac{(e\hbar)^2}{2} W_{n_0}(x_1, y_1, \dots, x_a, y_a) g_+(x_1, y_1) \cdots g_+(x_a, y_a).
\end{aligned} \tag{178}$$

When $a = 1$, (178) is, (dropping the non-periodic terms in W_{n_0}),

$$-i\hbar \int dx g_+(x_1, x) g_+(x, y_1) (\varphi'(x - y_1) - \varphi'(x - x_1)) = \frac{(e\hbar)^2}{2} p(x_1 - y_1) g_+(x_1, y_1), \tag{179}$$

and when $a = 2$ we have, using (179),

$$\begin{aligned}
& -i\hbar g_+(x_1, y_1) \int dx g_+(x_2, x) g_+(x, y_2) (\varphi'(x - y_1) - \varphi'(x - x_1)) - \\
& i\hbar g_+(x_1, y_1) \delta(y_2 - x_2) (\varphi'(y_1 - y_2) - \varphi'(x_1 - y_2)) + (x_1 \leftrightarrow x_2, y_1 \leftrightarrow y_2) = \\
& \frac{(e\hbar)^2}{2} (p(x_1 - y_2) + p(x_2 - y_1) - p(x_2 - x_1) - p(y_2 - y_1)) \times \\
& g_+(x_1, y_1) g_+(x_2, y_2).
\end{aligned} \tag{180}$$

When $x_1 = x_2$ and $y_1 = y_2$, (180) reduces to (179). Furthermore, when $a > 2$, (178) still contain terms of the same type as (180). Hence if (180) is satisfied, then (178) is satisfied for all a . In the case $m = 0$, an explicit solution to (179) might be found. By letting $m \rightarrow 0$ in the expression for g_+ , we have

$$g_+(x_1, y_1) = -\frac{1}{2\pi r} (i + \sum_{k \neq 0} e^{ik(x_1 - y_1)} \text{sgn}(k)).$$

Writing

$$\varphi(x_1 - y_1) = \sum_{k \neq 0} a_k (e^{ik(x_1 - y_1)} - 1),$$

where $a_k = a_{-k}$, we find the following solution to (179)

$$\varphi(x_1 - y_1) = \frac{e^2 \hbar}{4\pi r} \sum_{k \neq 0} \frac{\text{sgn}(k)}{k^3} (e^{ik(x_1 - y_1)} - 1). \tag{181}$$

This solution seems to satisfy (180) as well, even though we haven't really proved this. For a general m , (179) takes the form,

$$\begin{aligned}
& \sum_{k' (k - k' \neq 0)} \frac{(\hbar k' + im)(\hbar k' + im)}{E_k E_{k'}} (k - k') a_{k - k'} = \\
& \frac{e^2 \hbar}{4\pi r} \left(\sum_{k' (k - k' \neq 0)} \frac{\hbar k' + im}{E_{k'} (k - k')^2} - r^2 \frac{\pi^2}{3} \frac{\hbar k + im}{E_k} \right), \forall k.
\end{aligned} \tag{182}$$

We haven't found an analytic solution to (182) but it can be solved numerically. It remains to extend the ansatz to include also the non-periodic terms in W_{n_0} . We might get a hint on how to do this modification by calculating the ground state using ordinary time-independent perturbation theory. Writing the Hamiltonian as $\hat{H} = \hat{H}_0 + \hat{H}_I$ where

$$\hat{H}_0 = \int dx \left(-i\hbar \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial(x+\epsilon)} \gamma_{\alpha\beta} \hat{L}_{\alpha\beta}(x, x+\epsilon) + m \lim_{\epsilon \rightarrow 0} \gamma_{\alpha\beta}^0 \hat{L}_{\alpha\beta}(x, x+\epsilon) \right),$$

and

$$\hat{H}_I = \frac{1}{2} \int dx \hat{E}^2(x),$$

and regarding H_I as a perturbation, perturbation theory proceeds in the usual manner. Note that the groundstate of H_I is infinitely degenerate, each one of the states Ω_{n_0} given by (155) being as good as any other. However, to be able to do this, we need to know all excited states as well when $e = 0$. We will find these states presently.

4.3 Excited states

We expect a one-pair state to have the form $\hat{a}\Omega$ (at least when $e = 0$), where

$$\hat{a} = \int dx dy N_{\alpha\beta}(x, y) \hat{L}_{\alpha\beta}(x, y), \quad (183)$$

and

$$N_{\alpha\beta}(x, y) = A(x, y)\delta_{\alpha\beta} + B(x, y)\gamma_{\alpha\beta} + C(x, y)\gamma_{\alpha\beta}^0 + D(x, y)\gamma_{\alpha\beta}^1.$$

We want $\hat{a}\Omega$ to be an eigenstate of \hat{H} , with energy E' . Hence

$$[\hat{H}, \hat{a}]\Omega = E\hat{a}\Omega, \quad (184)$$

where $E = E' - E_{(0)}$. We also want $\hat{a}\Omega$ to be an eigenstate of \hat{P} , with momentum P , i.e.

$$[\hat{P}, \hat{a}]\Omega = P\hat{a}\Omega. \quad (185)$$

We obtain,

$$\begin{aligned} [\hat{H}, \hat{a}] = & \frac{(e\hbar)^2}{2} \int dx dy V(x, y) N_{\alpha\beta}(x, y) \hat{L}_{\alpha\beta}(x, y) + \\ & e\hbar \int dz dx dy \theta(x, y, z) N_{\alpha\beta}(x, y) \hat{L}_{\alpha\beta}(x, y) \hat{E}(z) - \\ & i\hbar \int dx dy \{ (\partial_x A(x, y) + \partial_y A(x, y)) \gamma_{\alpha\beta} + (\partial_x B(x, y) + \partial_y B(x, y)) \delta_{\alpha\beta} + \\ & (\partial_y C(x, y) - \partial_x C(x, y)) \gamma_{\alpha\beta}^1 + (\partial_y D(x, y) - \partial_x D(x, y)) \gamma_{\alpha\beta}^0 \} \hat{L}_{\alpha\beta}(x, y) + \\ & 2m \int dx dy (B(x, y) \gamma_{\alpha\beta}^1 + D(x, y) \gamma_{\alpha\beta}) \hat{L}_{\alpha\beta}(x, y), \end{aligned} \quad (186)$$

and

$$[\hat{P}, \hat{a}] = -i\hbar \int dx dy \{ (\partial_x + \partial_y) N_{\alpha\beta}(x, y) \} \hat{L}_{\alpha\beta}(x, y). \quad (187)$$

We realize that if there is any solution to (184) when $e \neq 0$, then it is state dependent. However, when $e = 0$, we can find state independent solutions to (184) and (185). Let us investigate this in detail. We find,

$$\begin{aligned} -i\hbar(\partial_x + \partial_y)A &= PA & -i\hbar(\partial_x + \partial_y)B &= PB \\ -i\hbar(\partial_x + \partial_y)C &= PC & -i\hbar(\partial_x + \partial_y)D &= PD \\ -i\hbar(\partial_x + \partial_y)A + 2mD &= EB & -i\hbar(\partial_x + \partial_y)B &= EA \\ -i\hbar(\partial_y - \partial_x)C + 2mB &= ED & -i\hbar(\partial_y - \partial_x)D &= EC. \end{aligned} \quad (188)$$

If (188) is satisfied, then (184) and (185) are satisfied. Solving (188) and demanding A, B, C and D to be $2\pi r$ -periodic functions of both x and y separately leads to,

$$\begin{aligned} A(x, y) &= P e^{\frac{i}{\hbar}(p_1 x + p_2 y)} & B(x, y) &= E e^{\frac{i}{\hbar}(p_1 x + p_2 y)} \\ C(x, y) &= \frac{(E^2 - P^2)(p_2 - p_1)}{2mE} e^{\frac{i}{\hbar}(p_1 x + p_2 y)} & D(x, y) &= \frac{(E^2 - P^2)}{2m} e^{\frac{i}{\hbar}(p_1 x + p_2 y)}, \end{aligned} \quad (189)$$

where

$$\begin{aligned} P &= p_1 + p_2 & E &= \pm E_1 \pm E_2 \\ E_1 &= \sqrt{p_1^2 + m^2} & E_2 &= \sqrt{p_2^2 + m^2} \end{aligned}$$

and $p_1 = \frac{\hbar n_1}{r}$, $p_2 = \frac{\hbar n_2}{r}$ for arbitrary integers n_1 and n_2 . Hence the one-pair states are labelled by p_1 and p_2 and we have recovered the two-particle interpretation of these states when $e = 0$. Acting with the so defined $\hat{a}(p_1, p_2)$ on the $e = 0$ ground state we find,

$$\begin{aligned} \hat{a}(p_1, p_2)\Omega &= i2\pi r \delta_{p_1, -p_2} \frac{E}{2m} (E + 2E_1)(\Omega^- - \Omega^+) + \\ &\Psi_{n_0} \frac{(E + E_1 + E_2)(E - E_1 + E_2)(E + E_1 + E_2)}{2E E_1 E_2} \times \\ &\sum_{a=1}^{\infty} \frac{1}{a!} \int d^a x d^a y \sum_{b=1}^a e^{\frac{i}{\hbar}(p_1 x_b + p_2 y_b)} \times \\ &\{(E_1 + E_2 + \frac{i}{m}(E_1 p_2 - E_2 p_1)) \prod_{j \neq b} g_+(x_j, y_j) \Psi_{(n_0, x_1, y_1, \dots, x_a, y_a)}^+ + \\ &(E_1 + E_2 - \frac{i}{m}(E_1 p_2 - E_2 p_1)) \prod_{j \neq b} g_-(x_j, y_j) \Psi_{(n_0, x_1, y_1, \dots, x_a, y_a)}^-\}. \end{aligned} \quad (190)$$

Thus we see explicitly that \hat{a} annihilates Ω when

$$E = -E_1 + E_2, \quad E = E_1 - E_2, \quad E = -E_1 - E_2,$$

which proves that we have found the correct ground state when $e = 0$. Denote the corresponding operators by $\hat{a}_{-+}(p_1, p_2)$, $\hat{a}_{+-}(p_1, p_2)$ and $\hat{a}_{--}(p_1, p_2)$. In the remaining case $E = E_1 + E_2$ the corresponding operator is denoted $\hat{a}_{++}(p_1, p_2)$ (which creates a pair from vacuum). Also,

$$\hat{a}_{++}^\dagger(p_1, p_2) = -\hat{a}_{--}(-p_2, -p_1), \quad \hat{a}_{+-}^\dagger(p_1, p_2) = -\hat{a}_{-+}(-p_2, -p_1).$$

Thus, when $e = 0$, an arbitrary excited state can be constructed by acting with products of \hat{a}_{++} :s on the ground state. We won't go into the complicated discussion of trying to solve (184) when $e \neq 0$ as it requires an explicit knowledge of the interacting ground state, which we don't have. Note however that (185) is solved independently of e .

5 Discussion

As we have seen, Dirac quantization works perfectly well for electrodynamics on a cylindrical space-time. Not finding the explicit ground state is a disappointment since comparing calculated expectation values found using our methods with previous results would have been interesting. However, we believe that with some more effort, the exact ground state in this formalism will be found. The formalism is easily extended to $3 + 1$ -dimensions and it should be interesting to investigate what happens in that case. The appearance of photons will necessarily complicate the discussion. Another generalization of this model is Yang-Mills coupled to fermions. The representation defined in this paper should, without too much effort, be generalizable to the non-abelian case. Finally, gravity coupled to fermions is a theory that should be investigated in this framework.

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References

- [1] A. Ashtekar, *Lectures on non-perturbative canonical gravity*, World Scientific (1991)
- [2] J. Schwinger, *Gauge invariance and mass. II*, Phys. Rev. **128** (1962) 2425-2429
- [3] N. S. Manton, *The Schwinger model and its axial anomaly*, Annals of Physics **159** (1985) 220-251
- [4] J.E. Hetrick and Y. Hosotani, *QED on a circle*, Phys. Rev. **D 38** (1988) 2621-2624
- [5] K. Sundermeyer, *Constrained Dynamics*, Lecture Notes in Physics 169, Springer Verlag (1982)
- [6] C. Rovelli and L. Smolin, *Loop representation for lattice gauge theory*, PRINT-90-0201 (SYRACUSE) (1990)
- [7] R. Floreanini and R. Jackiw, *Functional representation for fermionic quantum fields*, Phys. Rev. **D 37** (1988) 2206-2224
- [8] C. Rovelli and L. Smolin, *Loop space representation of quantum general relativity*, Nuclear Physics **B331** (1990) 80-152
- [9] A. Ashtekar and C. Rovelli, *A loop representation for the quantum Maxwell field*, Class. and Quant. Grav. **9** (1992) 1121-1150

- [10] R.S. Tate, *An algebraic approach to the quantization of constrained systems: finite dimensional examples*, Ph. D. thesis, Syracuse University (1992)